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ACTION OF A BURIED IMPULSIVE LOAD ON A VISCOELASTIC LAYER COVERING AN ELASTIC HALF-SPACE

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Considered is the plane problem of the action of an impulsive source of perturbations on a two-layer foundation consiting of an upper, viscoelastic layer fully coupled to an elastic half-space. The upper boundary of the foundation is load-free. The impulsive source is situated within the elastic half-space at a specified depth, and is defined in terms of the mass forces in the form of a product of delta functions of the coordinates and time. The investigation is carried out with the aid of Fourier and Laplace integral transforms. The displacements within the layer are written in the form of uniformly converging series in terms of the waves reflected within the layer. A viscoelastic, Boltzmann-type medium is considered as an example.

A viscoelastic layer of thickness R covers an elastic half-space. We study the dynamic behavior of the layer when an impulsive source of perturbations is switched on within the half-space at a distance $H - h$ from the boundary separating the layer and the half-space.

The problem can be reduced to that of solving the following Cauchy equations:

$$\sigma_{kl,l}^{(1)} = \rho^{(1)}\ddot{u}_k^{(1)}, \quad \sigma_{kl,l}^{(2)} + \rho^{(2)}X_k = \rho^{(2)}\ddot{u}_k^{(2)} \quad (k,\ l = 1,2) \tag{1}$$

in the layer and in the half-space, respectively, with zero initial conditions and the following conditions and the boundary surfaces:

$$z = 0, \quad \sigma_{kl}^{(1)} = 0; \quad z = h, \quad \sigma_{kl}^{(1)} = \sigma_{kl}^{(2)}, \quad u_k^{(1)} = u_k^{(2)} \tag{2}$$

where $\rho^{(j)}$ denote the densities of the media and $u_k^{(j)}$ are the displacements in the layer and half-space. The relations connecting the stresses and deformations are written in the form

$$\sigma_{kl} = \delta_{kl}L(\varepsilon) + 2M(\varepsilon_{kl}) \quad (\varepsilon = u_{k,k}, \quad \varepsilon_{kl} = {}^1/_2\,(u_{k,l} + u_{l,k})) \tag{3}$$

Here L and M denote linear operators which are, for the viscoelastic medium occupying the layer, either differential with respect to time and have constant coefficients, or integral with respect to time and have difference kernels. For the elastic medium of the half-space the operators L and M degenerate into the Lamé constants λ and μ.

In solving the problem we utilize the integral Laplace transforms with respect to time and coordinate z with parameters s and p, respectively, and the integral Fourier transform with respect to coordinate x with parameter k. The transformed quantities will be denoted by the same letters as the original quantities, and the parameters on which they depend will be shown.

The mass forces vary according to the following law:

$$X_1 = X = 0, \quad X_2 = Z = P\delta(x)\,\delta(z - H)\,\delta(t)$$

Using the relation (3), applying the integral transforms indicated above to the problem described by the second equation of (1) and conditions (2), (3) and taking into account the condition of radiation for the displacements and stresses in the half-space, we obtain the following formulas for the displacements in the half-space:

$$u_1^{(2)}(k, s; z) = -\frac{Pik}{2s^2\Delta}\{\Delta \exp[-\nu_a(H' - z)] - T_+^*(k, s)\,Q_1 \times \exp[-\nu_a(H' + z)] + 4\nu_a\nu_b g Q_3 \exp[-\nu_a(H' - \nu_b z]\} - \frac{P\nu_b}{2s^2\Delta}\{\Delta \exp[-\nu_b(H' - z)] + T_+^*(k, s)\,Q_2 \exp[-\nu_b(H' + z)] - 4k^2 g Q_4 \exp[-\nu_b H' - \nu_a z]\} \tag{4}$$

$$u_2^{(2)}(k, s; z) = -\frac{\nu_a}{ik}(u_p^{(2)} - u_{pp}^{(2)} + u_{sp}^{(2)}) - \frac{ik}{\nu_b}(u_s^{(2)} - u_{ss}^{(2)} + u_{ps}^{(2)}),$$

$$H' = H - 2h$$

$$\Delta = T_-\Delta_0, \quad g_s = 2k^2 + s^2 v_s^2, \quad g = 2k^2 + s^2 b^2$$

$$T_\pm^* = g^2 \pm 4k^2\nu_a\nu_b, \quad \nu_a^2 = k^2 + s^2 a^{-2}, \quad \nu_b^2 = k^2 + s^2 b^{-2}$$

$$T_\pm = g_s^2 \pm 4k^2\nu_p\nu_s, \quad \nu_p^2 = k^2 + s^2 v_p^2, \quad \nu_s^2 = k^2 + s^2 v_s^2$$

$$a^2 = \frac{\lambda + 2\mu}{\rho^{(2)}}, \quad b^2 = \frac{\mu}{\rho^{(2)}}, \quad v_p^2 = \frac{\rho^{(1)}}{L + 2M}, \quad v_s^2 = \frac{\rho^{(1)}}{M}$$

Expressions for Δ_0 and Q_i are very bulky and therefore omitted.

Derivation of the formulas (4) and all subsequent transformations and computations were carried out under the assumption that

$$v_p / v_s = b / a = \nu = \text{const}$$

and the latter yields the relation $v_p a = v_s b = \Theta$.

Each of the primary longitudinal and transverse waves arriving at the media interface from the half-space, excites a longitudinal and a transverse wave in the layer. To make it more convenient, we divide the displacement field in the layer into parts determined by the consecutive reflections of the waves from the layer boundaries and represented by uniformly converging series [1] (from now on we shall adopt, for convenience, the notation $u_1^{(1)} = u$, $u_2^{(1)} = w$, $u(k, s; z) = u_p + u_s$, $w(k, s; z) = u_p\nu_p / ik - u_s ik / \nu_s$).

Let us write the formulas for u_p only

$$u_p(k, s; z) = \sum_{m, n=0}^{\infty} a_{mn}^{+} \exp[-(n + 1)\nu_p h - m\nu_s h + \nu_p z] + \sum_{m, n=0}^{\infty} a_{mn}^{-} \exp[-(n - 1)\nu_p h - m\nu_s h - \nu_p z]$$

$$a_{mn}^{+} = \Omega(V_{mn} T_- p^+ - V_{m-2, n} T_+ s^+ - V_{m-1, n-1} 4k^2 g s^-)$$

$$
\begin{aligned}
&a_{mn}^{-} = \Omega\,(V_{m,\,n-2}T_{+}p^{-} - V_{m-2,\,n-1}T_{-}s^{-} + V_{m-1,\,n-1}4k^{2}gs^{+})\\
&p^{\pm} = B\tau_1 \pm \nu_b ikC\tau_2, \quad s^{\pm} = ikD\tau_1 \pm \nu_b A\tau_2\\
&A = 2k^2(\nu_p - \nu_a)(\Theta^2 - 1) + 2s^2\nu_a b^{-2}\Theta^2\\
&C = 2\,(k^2 - \nu_s\nu_a)\,(\Theta^2 - 1)\\
&B = 2k^2(\nu_s - \nu_b)(\Theta^2 - 1) + 2s^2\nu_b b^{-2}\Theta^2\\
&D = 2\,(k^2 - \nu_p\nu_b)\,(\Theta^2 - 1)\\
&\Omega = \frac{2Pik\Theta^2}{s^2\Delta}, \quad \tau_1 = \nu_a \exp[-\nu_a(H-h)], \quad \tau_2 = \exp[-\nu_b(H-h)]
\end{aligned}
$$

All V_{mn} are given by algebraic expressions which remain, in every case, bounded as $s \to \infty$ (in particular, $V_{00} = 1$), and vanish even if one of the indices m or n assumes a negative value. The intensity of the waves reflected from the layer boundaries decays rapidly, therefore the first order waves refracted into the layer and reflected from the boundary $z = 0$ represent the greatest interest. Assuming that m and n are both zero, we arrive at the following formulas describing the transformants of the displacements in the layer from the first order waves refracted into this layer:

$$
\begin{aligned}
&u\,(k, s; z) = a_{00}^{+} \exp[-\nu_p(h-z)] + b_{00}^{+}\exp[-\nu_s(h-z)] \qquad (5)\\
&w\,(k, s; z) = c_{00}^{+} \exp[-\nu_p(h-z)] + d_{00}^{+}\exp[-\nu_s(h-z)]\\
&a_{00}^{+} = \Omega T_{-}\,\{\nu_a B \exp[-\nu_a(H-h)] +\\
&\qquad \nu_b kiC\exp[-\nu_b(H-h)]\\
&b_{00}^{+} = \Omega T_{-}\nu_s\left\{\nu_a D\exp[-\nu_a(H-h)] + \frac{\nu_b}{ik}A\exp[-\nu_b(H-h)]\right.\\
&c_{00}^{+} = \frac{\nu_p}{ik}a_{00}^{+}, \quad d_{00}^{+} = -\frac{ik}{\nu_s}b_{00}^{+}
\end{aligned}
$$

The method of inverting the transformants can be illustrated on one of the terms of the vertical component of the displacement field (5). We denote this term by w_{sa} and first apply the inverse Fourier transformation

$$
w_{sa}(s; x, z) = \frac{2P\Theta^2}{\pi s^2}\int_{-\infty}^{\infty}\frac{k^2\nu_a(k^2-\nu_p\nu_b)(\Theta^2-1)}{\Delta_0}\exp\{f(k, s; x, z)\}\,dk \qquad (6)
$$

$$
f\,(k, s; x, z) = -\nu_a(H-h) - \nu_s(h-z) + ikx
$$

We compute the integral (6) and other similar integrals using an asymptotic method which is found efficient in investigating forces rapidly varying with time which act on the medium. To separate the large parameter, in terms of inverse powers of which the solution is expanded, we perform the following change of variables:

$$
k = bv_s s\xi = av_p s\xi \qquad (7)
$$

Using (7) we can write all integrals of the type (6) describing the displacement components in the canonical form

$$
I\,(s; x, z) = \int_{-\infty}^{\infty} F\,(s; x, z, \xi)\exp\{-s\Theta f(x, z, \xi)\}\,d\xi \qquad (8)
$$

The amplitude and phase functions in (8) have singularities on the complex ξ-plane, and the relative distribution of these singularities depends on the relation connecting the rates of propagation of the waves through the elastic half-space and in the visco-elastic layer. In what follows, we shall assume that

$$a > b > 1 / v_p > 1 / v_s$$

The roots of the equations $T_- = 0$ and $\Delta_0 = 0$ and all branch points of the integrand function are distributed along the imaginary axis. From each branch point we produce a cut to infinity parallel to the real axis Re ξ. Deforming the contour of integration, we can reduce the integral (8) to a sum of residues, and a contour integral bypassing the cust. Leaving aside the problem of computing the residues (which yield the static and the Rayleigh part of the displacement field), we shall consider a method of computing contour integrals which bypass the cuts and describe the propagation of the longitudinal and transverse waves. When the action of forces is sufficiently short, the pressure arising in the medium is mainly concentrated in the neighborhood of the fronts of the expanding waves. The assumption that $s \gg 1$ in (4), (5) and (8), enables us to investigate the displacements near the wavefronts and to apply the asymptotic methods to computing integrals of the type (8). Let us use the stationary phase method. A stationary point is determined from the condition $f'(x, z, \xi) = 0$, with the relation $x = (H - h) \operatorname{tg} \alpha_2 + (h - z) \operatorname{tg} \beta_1)$ taken into account

$$\xi_c = \frac{i \sin \beta_1}{b} = \frac{i \sin \alpha_2}{a\theta}$$

and is situated on the imaginary axis below the lowest branch point. The remaining displacement components were computed under the assumption that the following relation holds:

$$\frac{\sin \beta_1}{\sin \alpha_1} = \frac{\sin \beta_2}{\sin \alpha_2} = \nu$$

where α_2 and β_2 denote the angles of incidence of the longitudinal and transverse waves from the half-space onto the interface $z = h$ and α_1, β_1 are the angles of refraction of the longitudinal and transverse waves into the layer. The contour bypassing the cuts becomes a stationary contour $ABCDE$ (see Fig.1) defined by the equation

$$\operatorname{Im} [f (x, z, \xi)] = \operatorname{Im} [f (x, z, \xi_c)]$$

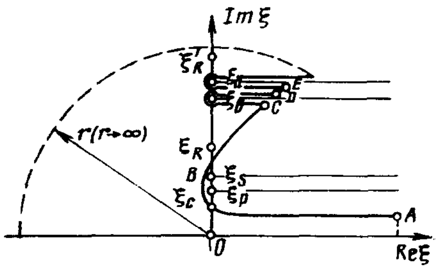

Fig. 1

When the condition $h / H \ll 1$ holds, the equation $f'(x, z, \xi) = 0$ has, in addition to ξ_c shown above, another solution ξ_c'. The second stationary point ξ_c' is situated on the imaginary axis near ξ_c. Let us now assume that the condition $h / H \ll 1$ does not hold, and restrict ourselves to the case when only one stationary point is present. The distribution of the points mentioned above and the stationary contour and shown in Fig. 1,and

$$\xi_R = \frac{i}{v_R}, \quad \xi_R{}' = \frac{i}{v_R{}'}, \quad \xi_b = \frac{i}{b}, \quad \xi_a = \frac{i}{a}, \quad \xi_s = \frac{i}{b\Theta}, \quad \xi_p = \frac{i}{a\Theta}$$

where v_R and $v_R{}'$ are the Rayleigh wave velocities in the elastic and viscoelastic media, respectively. The point ξ_a falls between ξ_b and ξ_s only when $\Theta^{-1} < \nu$.

Integration along the small circle surrounding each stationary point yields the major contribution towards the integral along the stationary contour $ABCDE$. In the cases when the stationary contour intersects the cut emerging from some point ξ_i the contribution of the integral at the stationary point ξ_c should be supplemented by the integral around the point ξ_i at which the initial integral of the type (8) assumes a sharply defined maximum value. Using the standard method given in [2] and retaining the principal term of the asymptotic expansion, we obtain

$$w_{sa}(x, z, s) = w_{sa}{}^{(1)}(x, z, \xi_c) + w_{sa}{}^{(2)}(x, z, \xi_a, \xi_b) \tag{9}$$

$$w_{sa}^{(1)}(x, z, \xi_c) = -P\sqrt{\frac{8b}{\pi}}\,\Phi(\xi_c)\,\frac{M + N\Theta}{\sqrt{G + R\Theta}}\,\frac{\Theta^2 - 1}{\Theta^2 s^2\sqrt{s\Theta}} \times$$

$$\exp\left\{-sf(x, z, \xi_c) - \frac{i\pi}{4}\right\} + O\left(\frac{1}{s^3}\right)$$

$$f(x, z, \xi_c) = \frac{(H - h)\sec\alpha_2}{a} + \frac{\Theta(h - z)\sec\beta_1}{b}, \quad N = \sin^2\beta_1$$

$$G = (h - z)\,\nu\cos^2\alpha_2\sec\beta_1, \quad R = (H - h)\cos^2\beta_1\sec\alpha_2$$

$$\Phi(\xi_c) = \cos^2\alpha_2\sin^2\beta_1\cos\beta_1\,\frac{\nu^{3/2}}{\Delta_0(\xi_c)}$$

$$M = [(1 - \nu^2\sin^2\alpha_2)(\nu^2 - \sin^2\beta_1)]^{1/2}$$

In the transformant (9) and in its analogs for the other displacement components Θ is the only variable dependent on the parameters of the Laplace transform. The transformants invert exactly when Θ and the time (parameter s) are connected by a simple relation, or invert approximately by expanding Θ into a series in inverse powers of s.

Let e.g. on elastoplastic layer be filled by a Boltzmann-type medium. Then

$$L(\varepsilon) = \lambda\varepsilon - \int_0^t Q_1(t - \tau)\,\varepsilon(\tau)\,d\tau, \quad M(\varepsilon_{kl}) = \mu\varepsilon_{kl} - \int_0^t Q_2(t - \tau)\,\varepsilon_{kl}(\tau)\,d\tau$$

and the volume and shear relaxation kernels are chosen in the form

$$Q_2(t) = \frac{\mu_0}{\tau_0}\exp\left\{-\frac{t}{\tau_0}\right\}, \quad Q_1(t) = \frac{\lambda}{\mu}\,Q_2(t)$$

The following time relations hold within the field of transformants;

$$\Theta = \frac{1}{1 - \eta}, \quad \eta = \eta_0\frac{1}{s\tau_0 + 1}, \quad \eta_0 = \frac{\mu_0}{\mu} < 1, \quad \rho = \frac{\rho^{(1)}}{\rho^{(2)}} = 1 \tag{10}$$

The exponential factor in (9) which contains Θ, can be written, with (10) taken into account, in the form

$$\exp\{-sf(x,z,\xi_c)\} = \exp\left\{-s\left[\frac{H-h}{a}\sec\alpha_2 + \frac{R-z}{b}\left(\sec\beta_1 + \frac{\eta_0}{2s\tau_0}\right)\right]\right\}\times$$
$$\left\{1 + \frac{h-z}{8b\tau_0}\,\frac{3\eta_0^2-4\eta_0}{s\tau_0+1} + O\left(\frac{1}{s^2}\right)\right\}$$

Expansions into series of the remaining expressions of (9) containing Θ retain terms of the order of s^{-1} inclusive. Applying the inverse Laplace transformation [3], we obtain the final formula for the displacement along the Oz -axis (the formula is given for $w_{sa}^{(1)}$ only)

$$w_{sa}^{(1)}(x,z,\tau) = P\sqrt{\frac{8b\tau_0}{\pi}}\,\frac{M+N}{\sqrt{G+R}}\,\frac{K}{\Delta_0'}\exp\left\{-\eta_0\omega - \frac{i\pi}{4}\right\}H(\tau-T)\times$$
$$\left\{2(\tau-T)^{3/2}\left[\Gamma\left(\frac{5}{2}\right)\right]^{-1}\exp[-(\tau-T)]_1F_1\left(\frac{3}{2},\frac{5}{2},\tau-T\right) + \right.$$
$$\left[\frac{\omega}{4}(3\eta_0-4) + \frac{N_1-G_1}{2} - \frac{E}{\Delta_0'}\right](\tau-T)^{5/2}\left[\Gamma\left(\frac{7}{2}\right)\right]^{-1}\times$$
$$\left.\exp[-(\tau-T)]_1F_1\left(\frac{3}{2},\frac{7}{2},\tau-T\right)\right\}$$

$$\omega = \frac{h-z}{b\tau_0}\sec\beta_1,\quad \tau = \frac{t}{\tau_0},\quad T = \frac{(H-h)\sec\alpha_2}{a\tau_0} + \frac{(h-z)\sec\beta_1}{b\tau_0}$$

$$N_1 = \frac{N}{M+N},\quad G_1 = \frac{G+2R}{2(G+R)},\quad K = \eta_0\nu^{3/2}\cos^2\alpha_2\sin^2\beta_1\cos\beta_1$$

$$E = M(1+4N\cos\beta_1\cos\alpha_2) - 2\nu N\cos^2\frac{\alpha_2}{2}\sqrt{1-\nu^2\sin^2\alpha_2} -$$
$$\cos\beta_1\left(\frac{\nu}{2}\cos\alpha_2 + 4N\sqrt{|\nu^2-\sin^2\beta_1|}\right)$$

$$\Delta_0' = (\cos\beta_1 + \sqrt{1-\nu^2\sin^2\alpha_2})(\nu\cos\alpha_2 + \sqrt{|\nu^2-\sin^2\beta_1|})$$

Here $H(\tau)$ is the Heaviside function and ${}_1F_1(\alpha,\beta,\tau)$ is a degenerate hypergeometric function. Taking into account the terms of the expansion of the order up to and including s^{-1}, we can introduce the following diminesionless parameters:

$$\frac{H}{b\tau_0} = H_0,\quad \frac{h}{b\tau_0} = h_0,\quad \frac{z}{b\tau_0} = z_0,\quad \frac{t}{\tau_0} = \tau,\quad \frac{\mu_0}{\mu} = \eta_0,\quad \frac{\rho^{(1)}}{\rho^{(2)}} = \rho$$

The properties of the viscoelastic medium and the type of the dynamic action, affect appreciably the parameters of the hypergeometric function which may, in some cases, degenerate into the elementary, Bessel and other functions. The formulas obtained using the above method describe the behavior of a viscoelastic medium near the fronts of the propagating waves,

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